Endurance Increase by Cyclic Control

Gottfried Sachs* and Theodoros Christodoulou†
Technical University of Munich, Munich, Federal Republic of Germany

Applying the minimum principle, maximum endurance flight is considered as an optimal cyclic control problem with a state variable constraint. It is shown that a significant increase in the maximum endurance can be achieved by dynamic flight having an optimal cyclic control when compared to the best steady-state flight. The optimal altitude range within the flight envelope is determined. Furthermore, it is shown that the powerplant type represents a key factor when compared to other aircraft characteristics. From this, it follows that turbojet-type engines have properties that can increase endurance via cyclic control. With regard to propeller-type powerplants, however, there appear to be no or only small improvements possible for the cyclic model considered here. It is also shown that an increase in the maximum lift/drag ratio improves the cyclic control efficiency, which is more enhanced than steady-state flight efficiency.

Nomenclature

C_D = drag coefficient
C_L = lift coefficient
D = drag
E = energy
G = function denoting constraint
g = acceleration due to gravity
H = Hamiltonian
h = altitude
J = performance criterion
K = lift-dependent drag factor, $C_D = C_{D_0} + KC_L^2$
L = lift
m = mass
n_v = exponent of thrust speed dependence, $T \sim V^{n_V}$
S = reference area
T = thrust
t = time
V = speed
γ = flight path angle
δ = throttle setting
λ, μ, ν = Lagrange multipliers
ρ = atmospheric density
σ = specific fuel consumption

Introduction

RECENTLY the application of cyclic control to aircraft cruise has become of great interest for improving fuel economy, since this type of cruise may offer possible ways of reducing fuel consumption in a given range.[1-12] In Refs. 1-12 it is shown that steady-state cruise is not generally optimal, but improvements may be achieved by nonsteady cruise with a cyclic control of thrust and lift. These papers deal with the *range* of the aircraft and its possibe increase by cyclic control. This paper is concerned with the *endurance* of the aircraft.‡ It is shown that a dynamic type of endurance flight with optimal cycle control can significantly increase maximum endurance when compared with the best values for steady-state flight.

Received June 18, 1984; revision received Dec. 19, 1984.

*Professor of Flight Mechanics and Flight Control. Associate Fellow AIAA.

†Research Assistant.

‡After this paper was written, Ref. 18 dealing with range and endurance became known to the authors.

Problem Formulation

The problem is to find periodic flight paths where the maximum endurance time per fuel consumed is greater than that for the best steady-state endurance flight. This is the equivalent of the periodic control problem involving the minimization of the following performance criterion:

$$J = -\frac{t_{\text{cyc}}}{m_f(t_{\text{cyc}})} \tag{1}$$

which represents the ratio of endurance time to fuel consumption. The performance criterion is subject to the dynamic system

$$\dot{V} = T/m - D/m - g\sin\gamma$$
$$\dot{\gamma} = (L/m - g\cos\gamma)/V$$
$$\dot{h} = V\sin\gamma$$
$$\dot{m}_f = \sigma T \tag{2}$$

The mass of the airplane can be considered constant, since the fuel consumed in one cycle is small when compared with the total of the mass, i.e.,

$$m_f(t_{\text{cyc}}) - m_f(0) \ll m \tag{3}$$

The periodicity of the flight path implies the following boundary conditions:

$$V(0) = V(t_{\text{cyc}}), \quad \gamma(0) = \gamma(t_{\text{cyc}}), \quad h(0) = h(t_{\text{cyc}}) \tag{4}$$

The initial condition for fuel consumption at $t = 0$ is given by

$$m_f(0) = 0 \tag{5}$$

The thrust, lift, and drag force models are

$$T = \delta T_{\max}(V,h), \quad L = C_L(\rho/2)V^2S, \quad D = C_D(\rho/2)V^2S \tag{6}$$

where

$$C_D = C_{D_0} + KC_L^2 \tag{7}$$

The atmospheric model used for air density ρ and the thrust dependence on altitude is based on the ICAO standard atmosphere.[13]

The control variables are the lift coefficient and throttle setting, subject to the following inequality constraints:

$$C_{L_{\min}} \leq C_L \leq C_{L_{\max}}, \qquad 0 \leq \delta \leq 1 \tag{8}$$

The specific fuel consumption is assumed to be a function of speed (or Mach number),

$$\sigma = \sigma(V) \tag{9}$$

The periodic control problem can now be stated as that of finding the control histories C_L and δ, the initial states [$V(0)$, $\gamma(0)$, $h(0)$], and the periodic cycle time t_{cyc} that minimize the performance criterion $J = -t_{cyc}/m_f(t_{cyc})$ subject to the dynamic system described by Eq. (2), the boundary conditions given by Eqs. (4) and (5), and the inequality constraints of Eq. (8) for the control variables.

Optimality Conditions

The first order necessary conditions for optimality can be determined by applying the minimum principle. For this purpose, the Hamiltonian is defined as

$$H = \lambda_V (T/m - D/m - g\sin\gamma) + \lambda_\gamma (L/m - g\cos\gamma)/V$$
$$+ \lambda_h V\sin\gamma + \lambda_{m_f}\sigma T \tag{10}$$

where the Lagrange multipliers $\lambda^T = (\lambda_V, \lambda_\gamma, \lambda_h, \lambda_{m_f})$ have been adjoined to the dynamic system of Eq. (2). The Lagrange multipliers are determined by§

$$\dot{\lambda}_V = \lambda_V (D_V - T_V)/m + \lambda_\gamma (L/m - VL_V/m - g\cos\gamma)/V^2$$
$$-\lambda_h \sin\gamma - \lambda_{m_f}(\sigma_V T + \sigma T_V)$$
$$\dot{\lambda}_\gamma = \lambda_V g\cos\gamma - \lambda_\gamma g\sin\gamma/V - \lambda_h V\cos\gamma$$
$$\dot{\lambda}_h = \lambda_V (D_h - T_h)/m - \lambda_\gamma L_h/(mV) - \lambda_{m_f}\sigma T_h$$
$$\dot{\lambda}_{m_f} = 0 \tag{11}$$

with boundary conditions

$$\lambda_V(0) = \lambda_V(t_{cyc}), \quad \lambda_\gamma(0) = \lambda_\gamma(t_{cyc}),$$
$$\lambda_h(0) = \lambda_h(t_{cyc}), \quad \lambda_{m_f}(t_{cyc}) = t_{cyc}/m_f^2(t_{cyc}) \tag{12}$$

The optimal controls C_L and δ are such that H is minimized. For this reason, C_L is determined either by (from $H_{C_L} = 0$)

$$C_L = \frac{1}{2K}\frac{\lambda_\gamma}{V\lambda_V} \tag{13}$$

or by the constraining bounds of Eq. (8). With regard to the throttle setting profile, δ can be considered to be of a bang-bang type since H is linear in δ. Thus,

$$\delta = 0 \quad \text{if} \quad H_\delta > 0 \qquad \delta = 1 \quad \text{if} \quad H_\delta < 0 \tag{14}$$

There may be a singular arc where the throttle setting takes on intermediate values, if $H_\delta = 0$ for a finite interval of time. However, this case was not observed in the numerical investigation.

Since the dynamic system described by Eq. (2) is autonomous, the Hamiltonian H is constant. For the case considered (i.e., free final time t_{cyc}), it is given by

$$H = 1/m_f(t_{cyc}) \tag{15}$$

§Partial derivatives of D,H,L,T and σ are denoted by subscripts, e.g., $D_V = \partial D/\partial V$.

Optimal Flight Paths with Altitude Constraints

In the numerical investigation, it was observed that the lower end of the optimal altitude range tended toward a value as small as possible. Therefore, it was necessary to introduce a lower bound for the admissible altitude range and to consider cyclic endurance flight as an periodic optimization problem with a state variable constraint $h \geq h_{\min}$. This case results in the additional conditions presented in the following paragraphs, based on Ref. 14.

Basically, two possibilities exist in regard to the constraint under consideration. One is characterized by the fact that the optimal flight path touches the constraint altitude boundary at only one point. The other possibility corresponds to a finite time for which the optimal flight path stays on the constraint boundary. Both possibililties have been observed in the numerical investigation, so that the additional conditions for each of them are presented in the following.

The altitude constraint can be formulated as

$$G(h) = h_{\min} - h \leq 0 \tag{16}$$

Since

$$G^{(1)}(V,\gamma) = -V\sin\gamma \tag{17a}$$

$$G^{(2)}(V,\gamma,h;T,C_L) = -\left(\frac{T}{m} - \frac{D}{m} - g\sin\gamma\right)\sin\gamma$$
$$-\left(\frac{L}{m}\right)\cos\gamma + g\cos^2\gamma \tag{17b}$$

the state variable constraint is of second order.

In regard to the first possibility, Eqs. (16) and (17a) yield the following conditions:

$$h(t_1) = h_{\min} \quad \text{and} \quad \gamma(t_1) = 0 \tag{18}$$

where t_1 denotes the time where the flight path touches the constraint altitude. At this point, λ_h shows a discontinuous change. Denoting by t_1^- the time just before the point under consideration and by t_1^+ immediately after, the following relation holds:

$$\lambda_h(t_1^+) = \lambda_h(t_1^-) - \nu_0 \tag{19}$$

where $\nu_0 \leq 0$.

The other possibility mentioned shows a constrained arc where the optimal flight path stays on the altitude boundary. On the constrained arc, the Hamiltonian is changed to

$$H \to H - \mu(t) G^{(2)}(V,\gamma,h;T,C_L) \tag{20}$$

Consequently, the following relations for the Lagrange multipliers exist:

$$\dot{\lambda}_V = -\frac{\partial H}{\partial V} + \mu(t)\frac{\partial}{\partial V}G^{(2)}$$

$$\dot{\lambda}_\gamma = -\frac{\partial H}{\partial \gamma} + \mu(t)\frac{\partial}{\partial \gamma}G^{(2)}$$

$$\dot{\lambda}_h = -\frac{\partial H}{\partial h} + \mu(t)\frac{\partial}{\partial h}G^{(2)} \tag{21}$$

The equation for λ_{m_f} remains unchanged since $G^{(2)}$ is independent of m_f.

The relation for $\mu(t)$ on the constrained arc is (from $H_{C_L} = 0$ and $\gamma = 0$)

$$\mu(t) = \lambda_V \frac{\partial C_D}{\partial C_L} - \frac{\lambda_\gamma}{V} \tag{22}$$

On the unconstrained arc, $\mu(t)=0$.

In regard to the optimal controls on the constrained arc, Eq. (17b) yields the following relation for C_L ($G^{(2)}=0$ with $\gamma=0$):

$$C_L = 2mg/(\rho V^2 S) \tag{23}$$

This represents the lift equation $L=mg$ for accelerated or decelerated horizontal flight. For δ as the other control variable, no contribution due to Eq. (17b) exists since $\gamma=0$.

In regard to the entry point t_1 of the constrained arc, Eqs. (16) and (17) yield the the following conditions:

$$h(t_1)=h_{\min},\ \gamma(t_1)=0,\quad \frac{1}{2K}\frac{\lambda_\gamma}{\lambda_V V}=\frac{2mg}{\rho V^2 S} \tag{24}$$

Some of the Lagrange multipliers show discontinuous changes at the entry point. Denoting by t_1^- the time just before the entry point and by t_1^+ immediately after, the following conditions hold (with ν_0 and ν_1 representing two additional unknowns):

$$\lambda_h(t_1^+)=\lambda_h(t_1^-)-\nu_0$$
$$\lambda_\gamma(t_1^+)=\lambda_\gamma(t_1^-)-\nu_1 V \tag{25}$$

λ_V and λ_{m_f} are continuous at the entry point.

In the numerical investigation, an optimization program based on the method of multiple shooting was applied.[15,16]

Basic Characteristics of Dynamic Endurance Flight with Optimal Cyclic Control

In Fig. 1, an optimal cycle for endurance maximization is shown in order to illustrate the basic characteristics of dynamic endurance flight. As indicated in this figure, an optimal cycle may be decomposed into two phases that can be characterized by the thrust behavior. In the first phase, the thrust is at its maximum, which (due to $T_{\max}>D$) results in an increase of the energy state of the aircraft. The second phase shows the thrust at its minimum (in the case considered here, $T=0$), where the energy added in the phase before is used to gain as much endurance as possible. Corresponding to the thrust behavior, the speed level in phase 1 is high compared to phase 2 and the altitude indicating the potential energy level shows an increase in phase 1 and a decrease in phase 2. The behavior just described can be used to give a physical insight into the reasons why dynamic endurance flight with optimal cyclic control offers an advantage. This is illustrated in Fig. 2, which shows the rate of specific energy added to the aircraft per fuel consumed, as

$$\frac{\dot{E}_{\text{add}}}{m_f(t_{\text{cyc}})}=\frac{TV/m}{m_f(t_{\text{cyc}})} \tag{26}$$

where $m_f(t_{\text{cyc}})$ represents the fuel used in one cycle. Due to the high speed level in phase 1, it is possible to reach high values in the energy rate per fuel consumed. This means a fuel utilization significantly better than that for the best steady-state flight, which is also shown in Fig. 2.

With regard to the rate of energy subtracted from the aircraft due to drag work

$$\dot{E}_{\text{sub}}=DV/m \tag{27}$$

Figure 3 shows that the drag work in phase 1 is increased as compared to steady-state flight. However, this effect is more than offset by the improved fuel utilization described above.

Powerplant Effects

The powerplant type must be considered to be a key factor in whether or not cyclic flight can provide an improvement in fuel efficiency. This is illustrated by a comparison of Fig. 4 and Fig. 1. Figure 1 illustrates the effects of a turbojet-type powerplant where the maximum thrust can be considered to be approximately constant as far as the speed changes during a cycle are concerned. The propulsion system applied in the example of Fig. 4 represents a propeller-type powerplant, which shows a decrease in the maximum thrust with speed. The

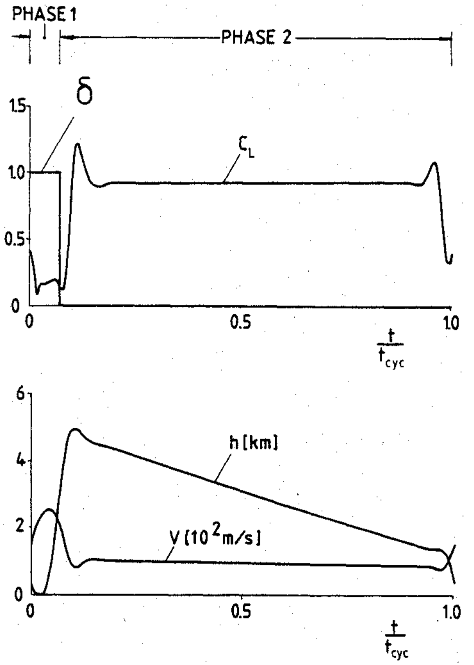

Fig. 1 Optimal cycle of dynamic endurance flight [$(T_{\max})_{h=0}=0.5$ mg, $n_V=0$, $t_{\text{cyc}}=11.87$ min] showing 63.5% endurance increase compared to the best steady-state flight.

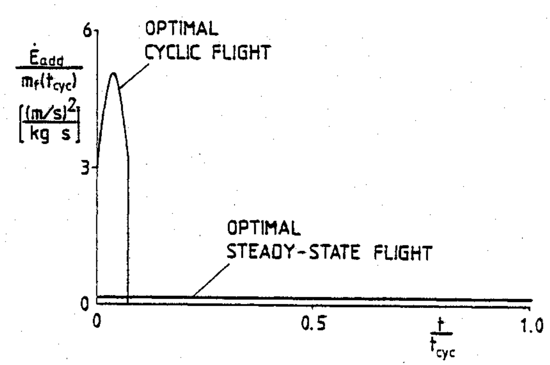

Fig. 2 Rate of energy added per fuel consumed for optimal cyclic and steady-state endurance flight, $(T_{\max})_{h=0}=0.5$ mg.

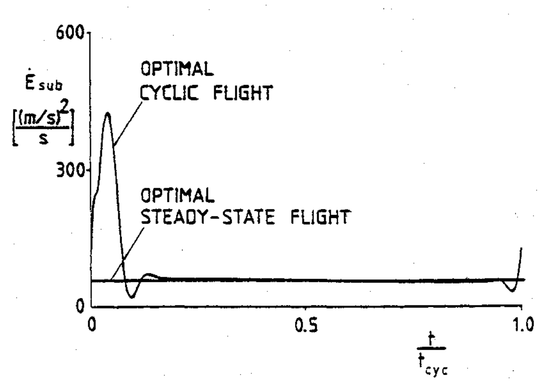

Fig. 3 Rate of energy subtracted for optimal cyclic and steady-state endurance flight.

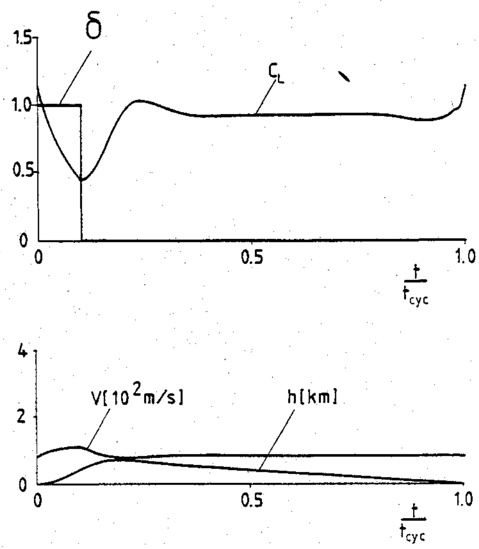

Fig. 4 Optimal cycle of dynamic endurance flight for a propeller-type powerplant ($n_V = -0.85$, $t_{cyc} = 2.73$ min) showing endurance increase of 0.46%.

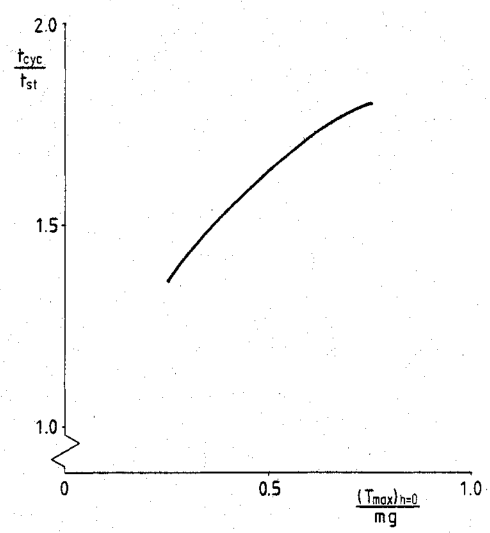

Fig. 5 Effect of maximum thrust/weight ratio on endurance increase.

thrust behavior is approximated by

$$T_{max} = (V/V_{ref})^{n_V} T_{ref}(h) \tag{28}$$

where the subscript "ref" denotes a reference value suitably chosen and n_V is used to characterize the powerplant type. The following values can be considered as typical[17]: $n_V \approx 0$ for turbojets, $n_V \approx -0.25$ for turbofans, and $n_V \approx -1.0$ for propeller-type powerplants. (In the latter situation, the propulsive power TV is constant.) According to the relation for the thrust behavior, specific fuel consumption is approximated by

$$\sigma = \sigma_{ref} / V^{n_V} \tag{29}$$

When compared with Fig. 1, Fig. 4 shows that the oscillatory character of the state variables in terms of the maximum amplitudes is significantly reduced, in particular with regard to speed. Accordingly, the improvement made possible by cyclic control is also reduced. As a result, powerplants having propellers (with $T \sim 1/V$) have properties such that an endurance increase via the cyclic control modeled here does not appear to be possible.

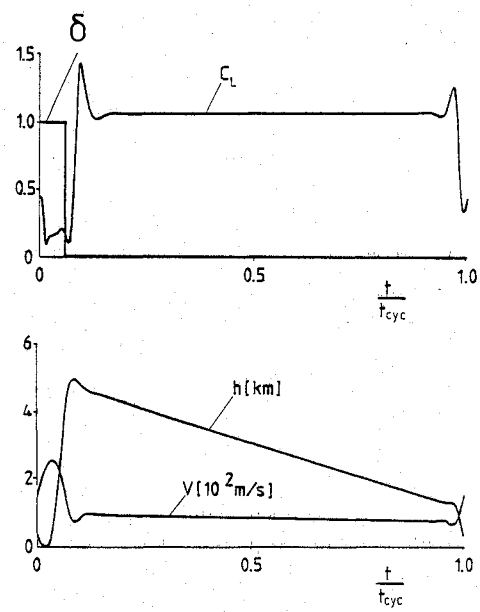

Fig. 6 Effect of maximum lift/drag ratio on optimal endurance flight. The 15% increase of $(L/D)_{max}$ compared to Fig. 1 yields 21.6% endurance increase of cyclic flight and 15.0% endurance increase of steady-state flight.

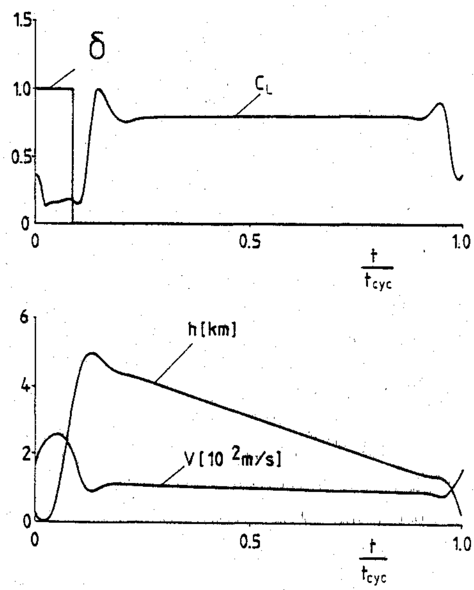

Fig. 7 Effect of maximum lift/drag ratio on optimal endurance flight. The 15% decrease of $(L/D)_{max}$ compared to Fig. 1 yields 20.1% endurance decrease of cyclic flight and 15.0% endurance decrease of steady-state flight.

For those powerplant types where cyclic control provides an improvement, the increase in maximum endurance is strongly dependent on the thrust level available. This is illustrated in Fig. 5, which shows the endurance time ratio t_{cyc}/t_{st} of optimal cyclic and steady-state flight as a function of maximum thrust/weight ratio (at $h = 0$), with the same amount of fuel assumed to be consumed in each case. From the results shown, it follows that the endurance increase can be substantially improved when more thrust is available. It also follows from Fig. 5 that there are already pronounced gains even for small thrust levels.

Another significant influence is due to the maximum lift/drag ratio $(L/D)_{max}$, since this quantity contributes to the maximum-thrust/minimum-drag ratio. Examples are presented in Figs. 6 and 7 where the maximum lift/drag ratio has been varied. Comparison with the results for the reference

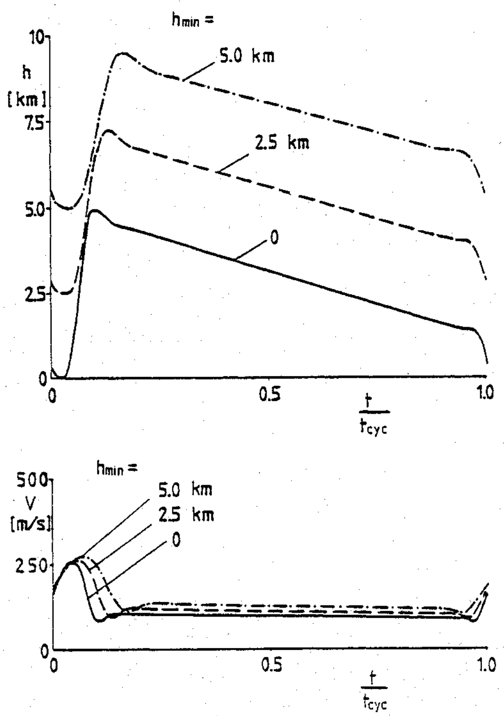

Fig. 8 Effect of an altitude constraint ($h \geq h_{min}$) on optimal cyclic endurance flight (state variables).

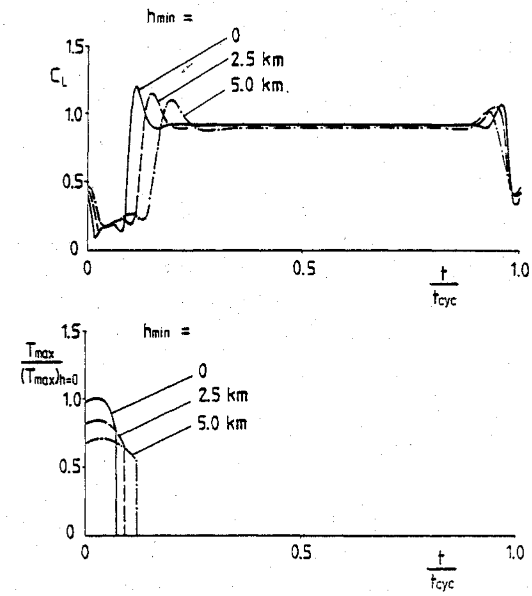

Fig. 9 Effect of an altitude constraint ($h \geq h_{min}$) on optimal cyclic endurance flight (control variables).

Table 1 Effect of altitude constraint on endurance increase

h_{min}, km	Endurance increase, %	Cycle time, min
0	63.5	11.87
2.5	51.4	10.14
5.0	40.2	8.51

value of $(L/D)_{max}$ used in Fig. 1 shows that an increase in the maximum lift/drag ratio improves the efficiency of cyclic control. It is interesting to note that this efficiency improvement is greater than in the case of steady-state flight, which also gains an advantage from an increase in $(L/D)_{max}$.

Altitude Effects

In the results presented so far, an altitude constraint $h \geq h_{min}$ had to be accounted for. For the aircraft model considered here this indicates that there is a tendency for the optimal altitude range to be as low as possible. In the following, it is shown how altitude affects maximum endurance and the optimal cycle when the altitude constraint h_{min} is changed. The altitude effects are of particular interest, since they show where the absolutely best value of maximum endurance is possible within the flight envelope of the aircraft and how this compares with greatest endurance achievable with steady-state flight.

An example is presented in Fig. 8, which shows the optimal histories of the state variables h and V for different values of the altitude constraint h_{min}. Endurance improvement is significantly reduced when the aircraft is forced to fly at a higher altitude range due to an increase of h_{min}, yielding the results presented in Table 1. The speed level is higher due to the density decrease. In regard to the control variables shown in Fig. 9, the thrust phase is increased relative to the gliding phase since the maximum thrust available decreases with altitude. The optimal lift coefficient shows changes mainly related to the thrust phase.

From the examples presented in Figs. 8 and 9 it follows that cyclic endurance flight yields the best values within the flight envelopes of the aircraft investigated. In the cases considered, cyclic flight should be performed at an altitude range of which the lower bound is given by $h = 0$.

Conclusions

In this paper, maximum endurance flight is considered as an optimal cyclic control problem with a state variable constraint given by a lower bound of the altitude. It is shown that the maximum endurance can be significantly increased by a dynamic type of endurance flight with an optimal cyclic control when compared with the best steady-state flight.

In regard to basic characteristics of optimal cyclic endurance flight, a cycle may be decomposed into two phases, of which one is a maximum-thrust increasing-energy condition and the other a zero-thrust decreasing-energy condition. It is shown that a better utilization of energy per fuel consumed is provided by cyclic control than in the case of steady-state flight. Furthermore, it is shown that the powerplant type represents a key factor when compared with other aircraft characteristics. From this it follows that turbojet-type engines have properties that allow optimal cyclic flight to increase the maximum endurance. With regard to propeller-type powerplants, however, no or only small improvements appear to be possible for the cyclic model considered here.

For those powerplant systems where cyclic control offers an advantage, the improvements achievable increase when more thrust is available, i.e., when the thrust/weight ratio is increased. In addition, it is shown that a better maximum lift/drag ratio also improves cyclic endurance flight efficiency, which is enhanced more than in the case of steady-state flight.

In regard to altitude effects, it is shown that, for the aircraft model considered, cyclic control yields its maximum endurance at an altitude range in which the lower bound is given by $h = 0$. This represents the best endurance achievable within the flight envelope.

Acknowledgment

The authors would like to thank Prof. Dr. R. Bulirsch of the Technische Universität München and Prof. Dr. H.J. Oberle of the University of Hamburg for making available the optimization program BOUNDSCO.

References

[1]Speyer, J.L., "On the Fuel Optimality of Cruise," *Journal of Aircraft,* Vol. 10, 1973, pp. 763-765.

[2]Speyer, J.L., "Nonopotimality of the Steady-State Cruise for Aircraft," AIAA Journal, Vol. 14, 1976, pp. 1604-1610.

[3]Speyer, J.L., Dannemiller, D. and Walker, D., "Periodic Optimal Cruise of a Hypersonic Vehicle," AIAA Paper 80-1777, 1980.

[4]Speyer, J.L., Dannemiller, D. and Walker, D., "Periodic Control of an Atmospheric Vehicle," *Collection of Papers, 25th Israel Annual Conference on Aviation and Astronautics,* 1983, pp. 245-255.

[5]Gilbert, E.G. and Parsons, M.G., "Periodic Control and the Optimality of Aircraft Cruise," *Journal of Aircraft,* Vol. 13, 1976, pp. 828-830.

[6]Gilbert, E.G., "Vehicle Cruise: Improved Fuel Economy by Periodic Control," *Automatica,* Vol. 12, 1976, pp. 159-166.

[7]Breakwell, J.V. and Shoee, H., "Minimum Fuel Flight Paths for Given Range," AIAA Paper 80-1660, 1980.

[8]Vinh, N.X., *Optimal Trajectories in Atmospheric Flight,* Elsevier, Amsterdam/Oxford/New York, 1981.

[9]Houlihan, S.C., Cliff, E.M. and Kelly, H.J., "Study of Chattering Cruise," *Journal of Aircraft,* Vol. 19, 1982, pp. 119-124.

[10]Grimm, W., "Numerische Berechnung kostenoptimaler Flugbahnen unter periodischen Randbedingungen," Diplomarbeit, Institut für Mathematik der Technischen Universität München, FRG, 1983.

[11]Walker, B.K., "Aircraft Fuel Economy Optimization by Periodic Control," M.S. Thesis, Massachusetts Institute of Technology, Cambridge, 1977.

[12]Schultz, R. and Zagalsky, N.R., "Aircraft Performance Optimization," *Journal of Aircraft,* Vol. 9, 1972, pp. 108-114.

[13]*ICAO Standard Atmosphere,* International Civil Aviation Organization, Montreal, 1964.

[14]Bock, H.G., "Numerische Behandlung von zustandsbeschränkten und Chebychef-Steuerungsproblemen," Course R1.06, Carl-Cranz-Gesellschaft, Oberpfaffenhofen, FRG, 1983.

[15]Oberle, HJ., "Numerische Berechnung optimaler Steuerungen von Heizung und Kühlung für ein realistisches Sonnenhausmodell," Institut für Mathematik der Technischen Universität München, FRG, Rept. TUM-M8310, 1983.

[16]Bulirsch, R., "Die Mehrzielmethode zur numerischen Lösung von nichtlinearen Randwertproblemen und Aufgaben der optimalen Steuerung," Rept. of the Carl-Cranz-Gesellschaft, Oberpfaffenhofen, FRG, 1971.

[17]Brüning, G. and Hafer, X, *Flugleistungen,* Springer-Verlag, Berlin/Heidelberg/New York, 1978.

[18]Lyons, D.T., "Improved Aircraft Cruise by Periodic Control," Ph.D. Dissertation, University of Michigan, Ann Arbor, 1980.